

The 2nd Workshop on QCD Matter under Rotation and Magnetic Fields

Huainan, Anhui

The quasinormal mode analysis of MHD

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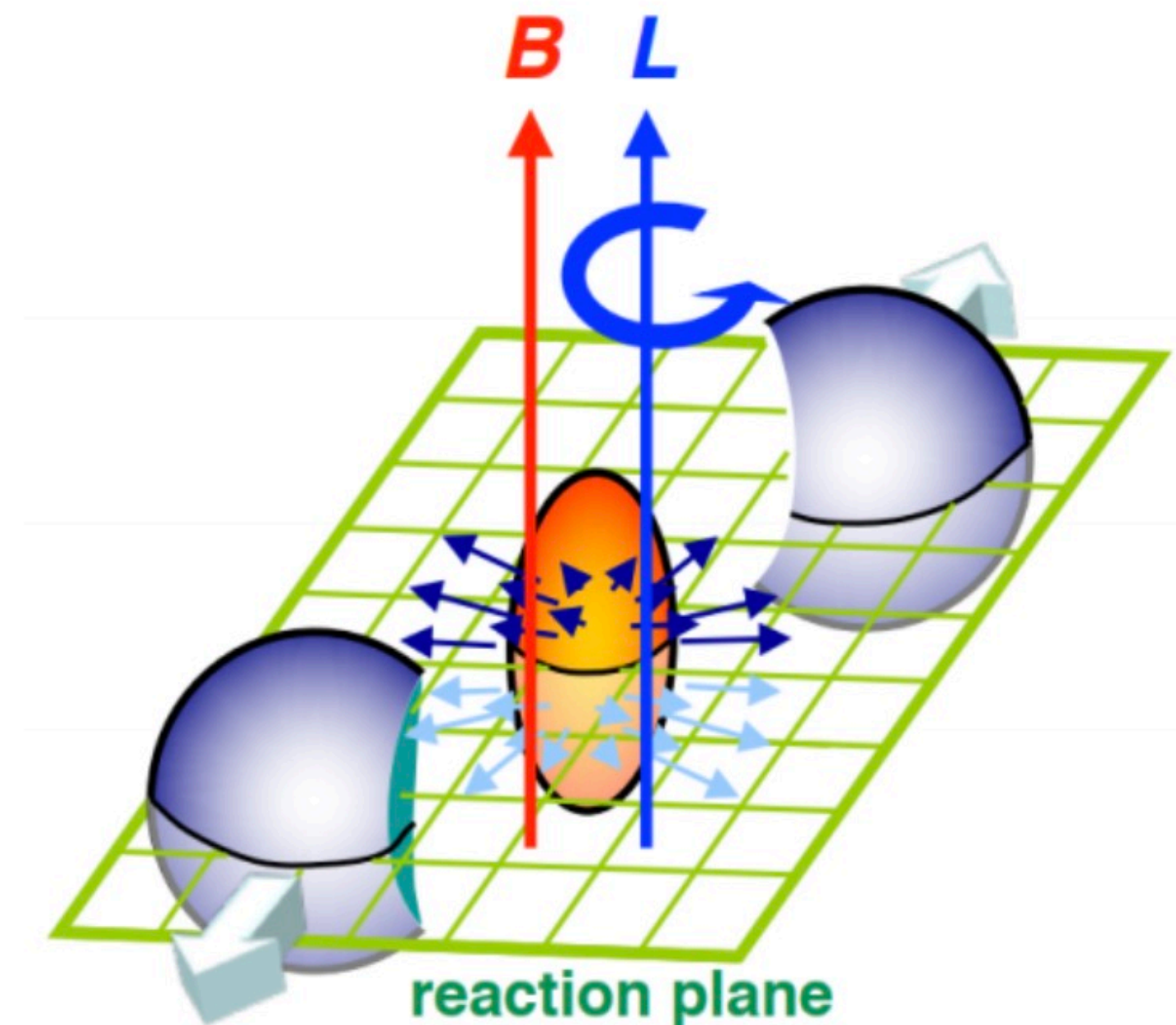
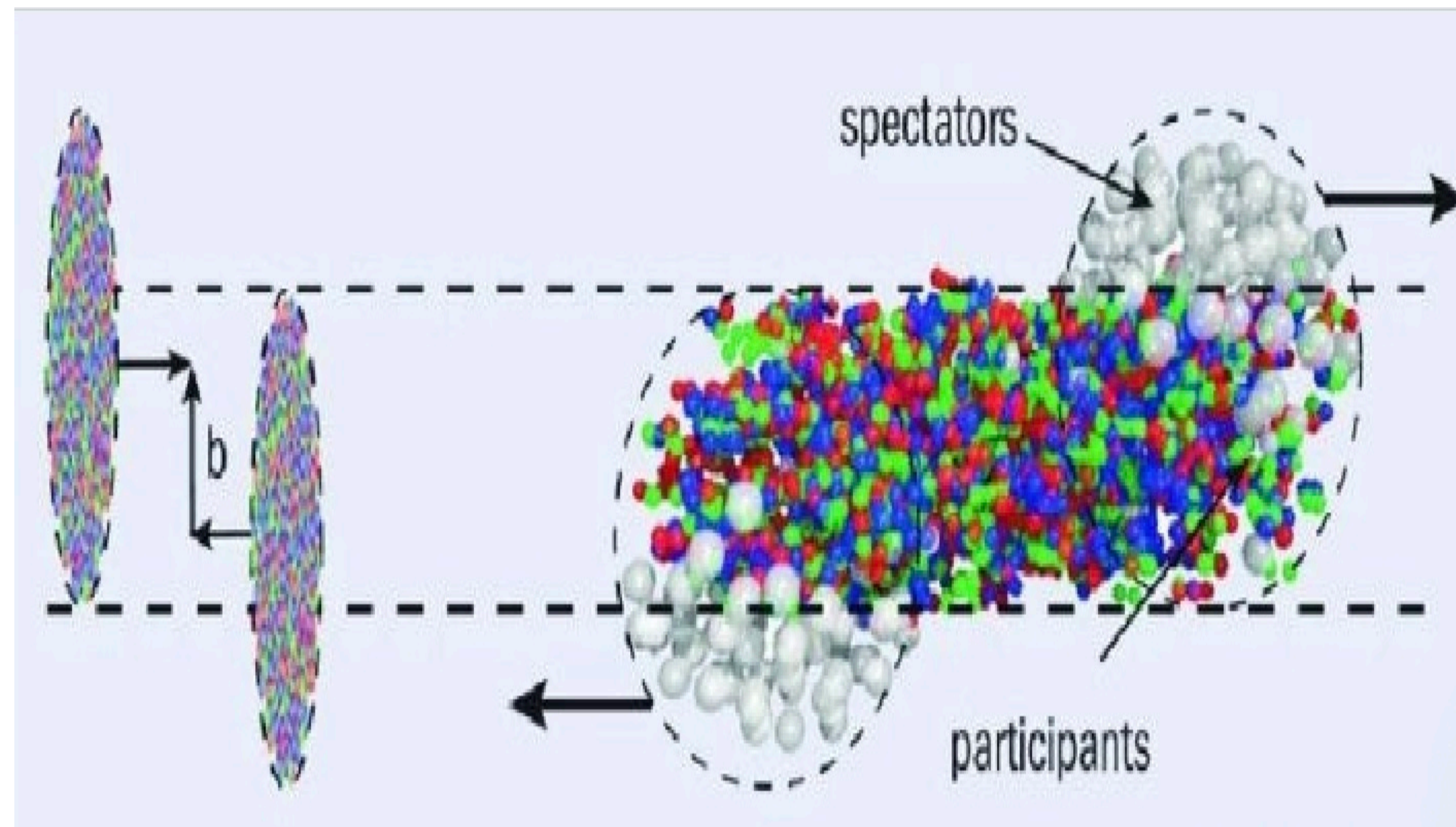
Ref: Zhe Fang, Koichi Hattori, **Jin Hu**, *Phys.Rev.D* 110 (2024) 5, 056049

Zhe Fang, Koichi Hattori, **Jin Hu**, *Phys.Rev.D* 112 (2025) 7, 076016

Outline

- Introduction
- First order MHD analysis
- IS-MHD analysis
- Summary and Outlook

Introduction

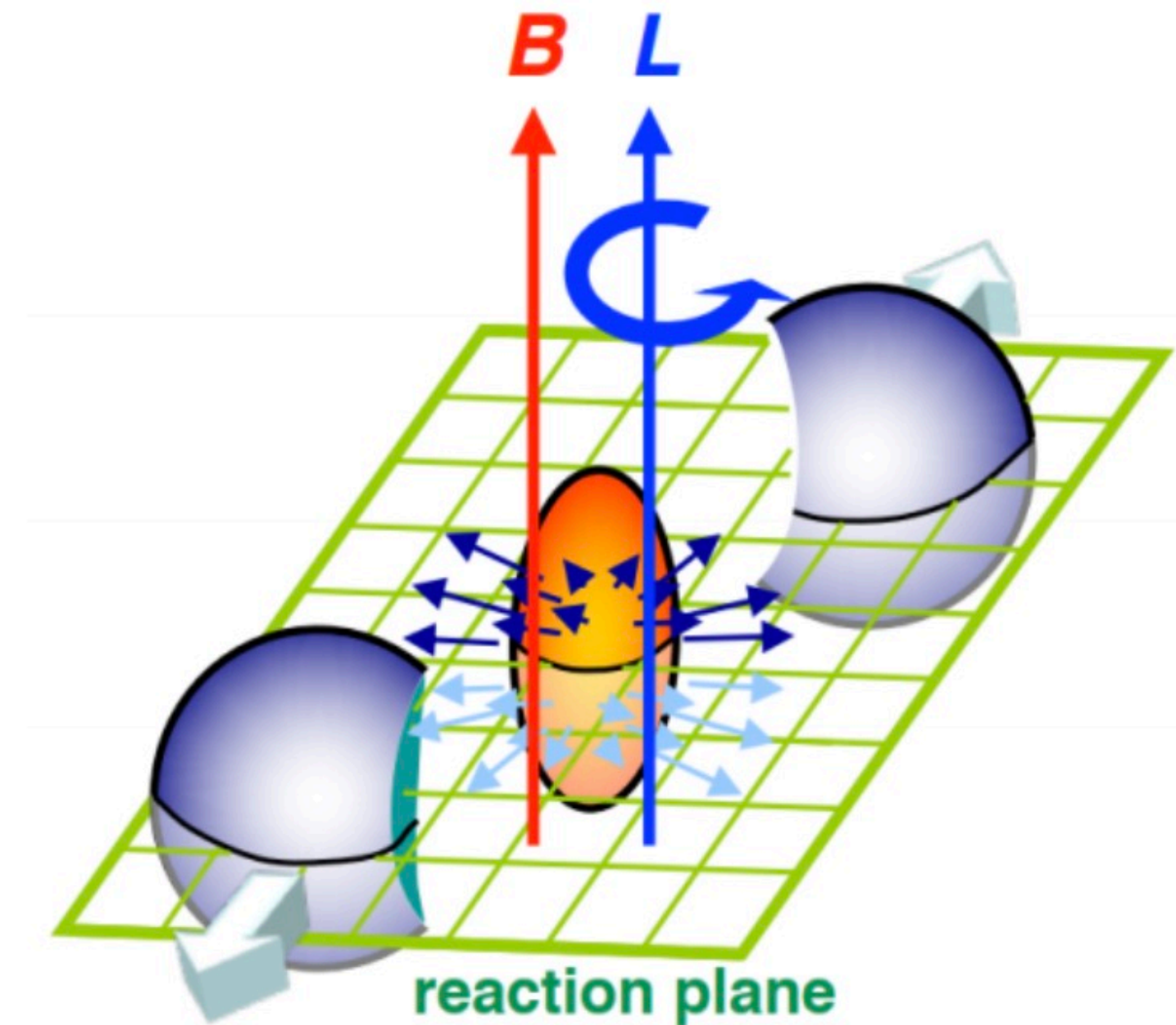
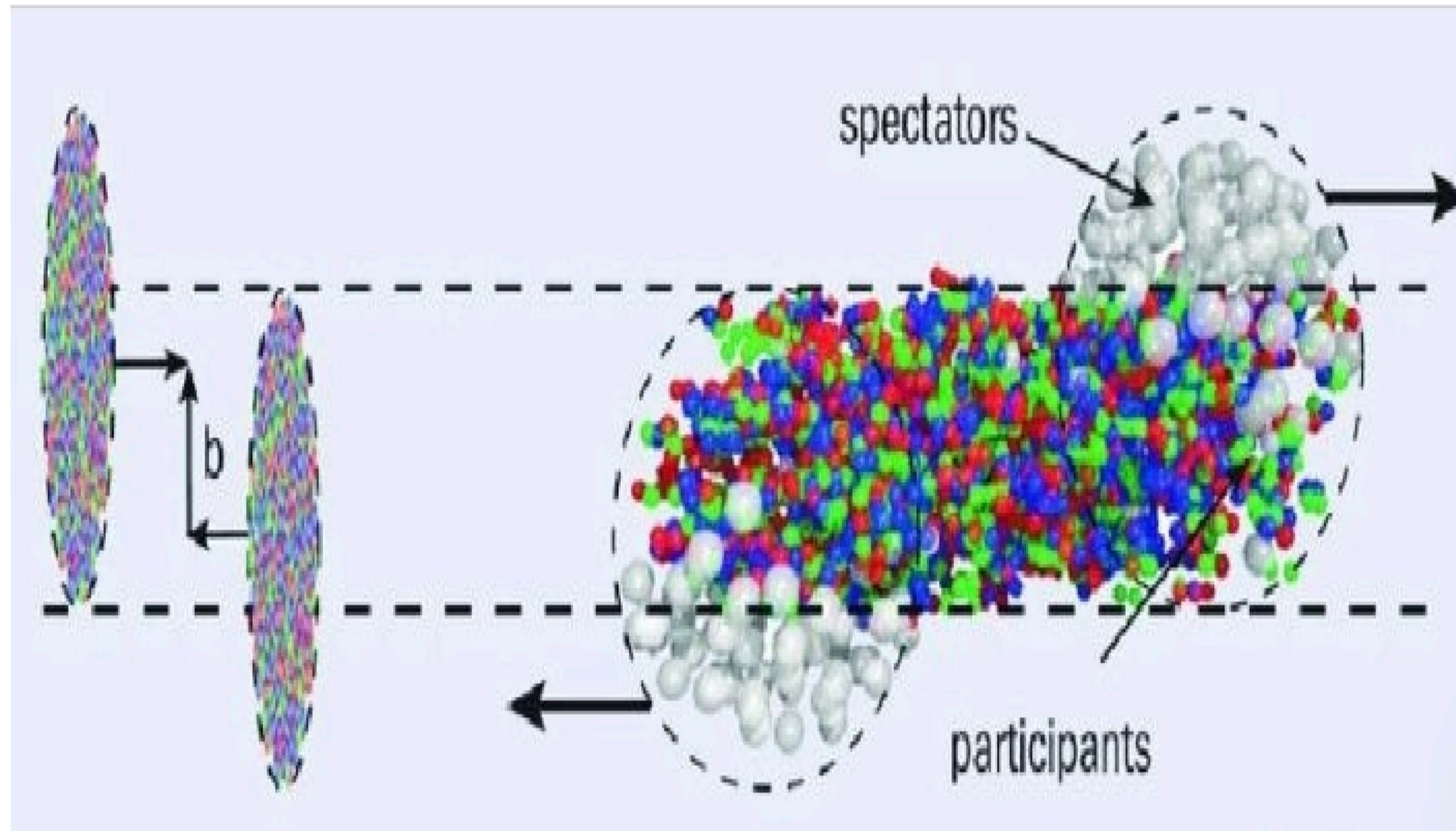


Non-central relativistic heavy-ion collisions can generate magnetic fields as high as $eB \sim 10^{13} - 10^{14} T$.

W.T. Deng, X.G.Huang, *Phys. Rev. C* 85, 044907 (2012) ...

It is essential to study the effects of magnetic fields on the physical properties and transport behavior of quark matter / QGP. One of the topics in this workshop.

Introduction



Non-central relativistic heavy-ion collisions can generate magnetic fields as high as

$$eB \sim 10^{13} - 10^{14} T.$$

W.T. Deng, X.G.Huang, *Phys. Rev. C* 85, 044907 (2012) ...

We focus on the influence of magnetic fields on hydrodynamics. In other words, we discuss magnetohydrodynamics (MHD), see e.g., K.Hattori, M.Hongo, X.G.Huang, *Symmetry* 14, 1851 (2022) .

Introduction

Picture from Web



Though it remains debated whether magnetic fields produce significant effects before their decay, [Z. Wang, J. Zhao, C. Greiner, Z. Xu, and P. Zhuang, Phys. Rev. C 105, L041901 \(2022\)](#), it is indisputable that MHD has been widely applied in astrophysics and cosmology, e.g., black hole accretion, supernova explosions, and binary neutron star mergers all rely on MHD simulations !

1st order MHD

Consider a neutral fluid, we have two basic equations

$$\partial_\mu \Theta^{\mu\nu} = 0 \quad \text{energy-momentum conservation}$$

and $\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$

magnetic flux conservation / Bianchi identity

Note that the electric field is not hydrodynamic and has damped out in the hydrodynamic limit (long time).

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad \text{also known as one-form symmetry}$$

$$Q(\Sigma_2) = \int_{\Sigma_2} \star F = \int_{\Sigma_2} \mathbf{B} \cdot d\mathbf{S} . \quad \text{J. Armas, A. Jain, } JHEP \text{ 01 (2020) 041}$$

1st order MHD

Consider a neutral fluid, we have two basic equations

$$\partial_\mu \Theta^{\mu\nu} = 0 \quad \text{0-form symmetry}$$

continuity eq $\partial_t \Theta^{t\nu} + \partial_i \Theta^{i\nu} = 0$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad \text{1-form symmetry}$$

extended continuity eq $\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$

In the sense of hydrodynamic limit,

$$\nabla \times (\dots), \quad \nabla \cdot (\dots) \rightarrow 0$$

1st order MHD

$$\partial_\mu \Theta^{\mu\nu} = 0$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

According to Lorentz symmetry, $SO(3,1) \xrightarrow{u^\mu} SO(3) \xrightarrow{B^\mu} SO(2)$

$$\Theta^{\mu\nu} = eu^\mu u^\nu + p_\parallel b^\mu b^\nu + p_\perp \Xi^{\mu\nu} + \Theta_{(1)}^{\mu\nu},$$

$$\tilde{F}^{\mu\nu} = B^\mu u^\nu - B^\nu u^\mu + \tilde{F}_{(1)}^{\mu\nu}$$

with $b^\mu \equiv B^\mu / \sqrt{B^\nu B_\nu}, \quad \Xi^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu - b^\mu b^\nu$

1st law of thermodynamics H_μ is the magnetic field

$$Ts = e + p - H_\mu B^\mu, \quad Tds = de - H_\mu dB^\mu$$

2nd law of thermodynamics

$$\partial_\mu s^\mu \geq 0, \quad s^\mu = su^\mu + s_{(1)}^\mu$$

1st order MHD

$$\partial_\mu \Theta^{\mu\nu} = 0 \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0$$

According to Lorentz symmetry,

$$\Theta^{\mu\nu} = eu^\mu u^\nu + p_\parallel b^\mu b^\nu + p_\perp \Xi^{\mu\nu} + \Theta_{(1)}^{\mu\nu},$$

$$\tilde{F}^{\mu\nu} = B^\mu u^\nu - B^\nu u^\mu + \tilde{F}_{(1)}^{\mu\nu}$$

2nd law of thermodynamics $\partial_\mu s^\mu = 0$ gives

$$p_\perp = p, \quad (p_\parallel - p_\perp)b^\nu + H^\nu B = 0 \quad H_\mu = -\mu_m^{-1} B_\mu$$

The zeroth order MHD is established.

Z.Fang, K. Hattori, J. Hu, *Phys.Rev.D* 110 (2024) 5, 056049

1st order MHD

Z.Fang, K. Hattori, J. Hu, *Phys.Rev.D* 110 (2024) 5, 056049

$$\Theta^{\mu\nu} = eu^\mu u^\nu + p_{\parallel} b^\mu b^\nu + p_{\perp} \Xi^{\mu\nu} + \Theta_{(1)}^{\mu\nu}, \quad \tilde{F}^{\mu\nu} = B^\mu u^\nu - B^\nu u^\mu + \tilde{F}_{(1)}^{\mu\nu}$$

2nd law of thermodynamics $\partial_\mu s^\mu \geq 0$ gives

$$\Theta_{(1)}^{\mu\nu} = -T\eta^{\mu\nu\rho\sigma}\partial_{(\rho}\beta_{\sigma)}, \quad \tilde{F}_{(1)}^{\mu\nu} = -T\rho^{\mu\nu\rho\sigma}\partial_{[\rho}(\beta H_{\sigma]})$$

with $X^{(\mu\nu)} \equiv \frac{X^{\mu\nu} + X^{\nu\mu}}{2}, \quad X^{[\mu\nu]} \equiv \frac{X^{\mu\nu} - X^{\nu\mu}}{2}$

$$\eta^{\mu\nu\rho\sigma} = (b^\mu b^\nu \quad \Xi^{\mu\nu}) \begin{pmatrix} \zeta_{\parallel} & \zeta_{\times} \\ \zeta'_{\times} & \zeta_{\perp} \end{pmatrix} \begin{pmatrix} b^\rho b^\sigma \\ \Xi^{\rho\sigma} \end{pmatrix} \quad (10a)$$

$$\begin{aligned} & + 2\eta_{\parallel} (b^\mu \Xi^{\nu(\rho} b^{\sigma)} + b^\nu \Xi^{\mu(\rho} b^{\sigma)}) \\ & + \eta_{\perp} (\Xi^{\mu\rho} \Xi^{\nu\sigma} + \Xi^{\mu\sigma} \Xi^{\nu\rho} - \Xi^{\mu\nu} \Xi^{\rho\sigma}), \\ \rho^{\mu\nu\rho\sigma} & = -2\rho_{\perp} (b^\mu \Xi^{\nu[\rho} b^{\sigma]} - b^\nu \Xi^{\mu[\rho} b^{\sigma]}) + 2\rho_{\parallel} \Xi^{\mu[\rho} \Xi^{\sigma]\nu}. \end{aligned} \quad (10b)$$

1st order MHD analysis

$$\begin{aligned}\partial_\mu \Theta^{\mu\nu} &= 0 & \Theta^{\mu\nu} &= eu^\mu u^\nu + p_\parallel b^\mu b^\nu + p_\perp \Xi^{\mu\nu} + \Theta_{(1)}^{\mu\nu}, \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 & \tilde{F}^{\mu\nu} &= B^\mu u^\nu - B^\nu u^\mu + \tilde{F}_{(1)}^{\mu\nu}\end{aligned}$$

Perturb the system from a static equilibrium,

$$e \rightarrow e + \delta e(x^\mu), \quad u^\mu \rightarrow u^\mu + \delta u^\mu(x^\mu), \quad B^\mu \rightarrow B^\mu + \delta B^\mu(x^\mu)$$

with $u^\mu = (1, 0, 0, 0)$, $B^\mu = (0, 0, 0, B)$

then Fourier transform to the complex space

$$\delta e(t, x, z) = \delta \tilde{e}(\omega, k_\perp, k_\parallel) e^{-i\omega t + ik_\perp x + ik_\parallel z} \dots$$

➡ two distinct sectors of equations for the fluctuations.

1st order MHD analysis

Alfven sector

$$\begin{pmatrix} A_{(0)} + iA_{(1)} \end{pmatrix} \begin{pmatrix} \delta B_y \\ \delta u_y \end{pmatrix} = 0, \quad k_{\parallel} = k \cos \theta, \quad k_{\perp} = k \sin \theta.$$

$$v_A = \frac{B}{\sqrt{e + p}}$$

$$A_{(0)} = \begin{pmatrix} \omega & Bk_{\parallel} \\ h \frac{v_A^2}{B} k_{\parallel} & h\omega \end{pmatrix}, \quad A_{(1)} = \begin{pmatrix} \rho'_{\perp} k_{\parallel}^2 + \rho'_{\parallel} k_{\perp}^2 & 0 \\ 0 & \eta_{\parallel} k_{\parallel}^2 + \eta_{\perp} k_{\perp}^2 \end{pmatrix}$$

coupled eqs for transverse fluctuations

$$\omega = \pm \sqrt{v_A^2 k_{\parallel}^2 - \frac{1}{4}(\tilde{\rho} - \tilde{\eta})^2 k^4} - \frac{i}{2}(\tilde{\rho} + \tilde{\eta})k^2$$

where

$$\tilde{\rho} = \rho_{\perp} \cos^2 \theta + \rho_{\parallel} \sin^2 \theta \quad \tilde{\eta} = \frac{\eta_{\parallel} \cos^2 \theta + \eta_{\perp} \sin^2 \theta}{e + p}$$

1st order MHD analysis

Alfven sector $\omega = \pm \sqrt{v_A^2 k_{\parallel}^2 - \frac{1}{4}(\tilde{\rho} - \tilde{\eta})^2 k^4} - \frac{i}{2}(\tilde{\rho} + \tilde{\eta})k^2$

first small k expansion, then $\theta \rightarrow \frac{\pi}{2}$

$$\omega = -\frac{i}{2}(\tilde{\rho} + \tilde{\eta})k^2 + O(k^3)$$

S. Grozdanov, D. M. Hofman, and N. Iqbal, *PRD* (2017)

J. Armas and F. Camilloni, *JCAP* (2022)

first $\theta \rightarrow \frac{\pi}{2}$, $\omega(\theta = \frac{\pi}{2}) = -i \frac{\eta_{\perp}}{e+p} k^2, \quad -i \rho_{\parallel} k^2$

$\theta \rightarrow \frac{\pi}{2}$ and $k \rightarrow 0$ can't commute. Singular perturbation problem !

$$\omega(\theta \rightarrow \frac{\pi}{2}) = -\frac{i}{2}(\rho_{\parallel} + \frac{\eta_{\perp}}{e+p})k_{\perp}^2 \pm \lim_{\theta \rightarrow \pi/2} \sum_{n=1}^{\infty} \frac{c_n(\rho_{\parallel} - \frac{\eta_{\perp}}{e+p})^{2n}}{(v_A \cos \theta)^{2n-1}} k_{\perp}^{2n+1}$$

1st order MHD analysis

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Alfven sector $\omega = \pm \sqrt{v_A^2 k_{\parallel}^2 - \frac{1}{4}(\tilde{\rho} - \tilde{\eta})^2 k^4} - \frac{i}{2}(\tilde{\rho} + \tilde{\eta})k^2$

A remedy for singular perturbation problem

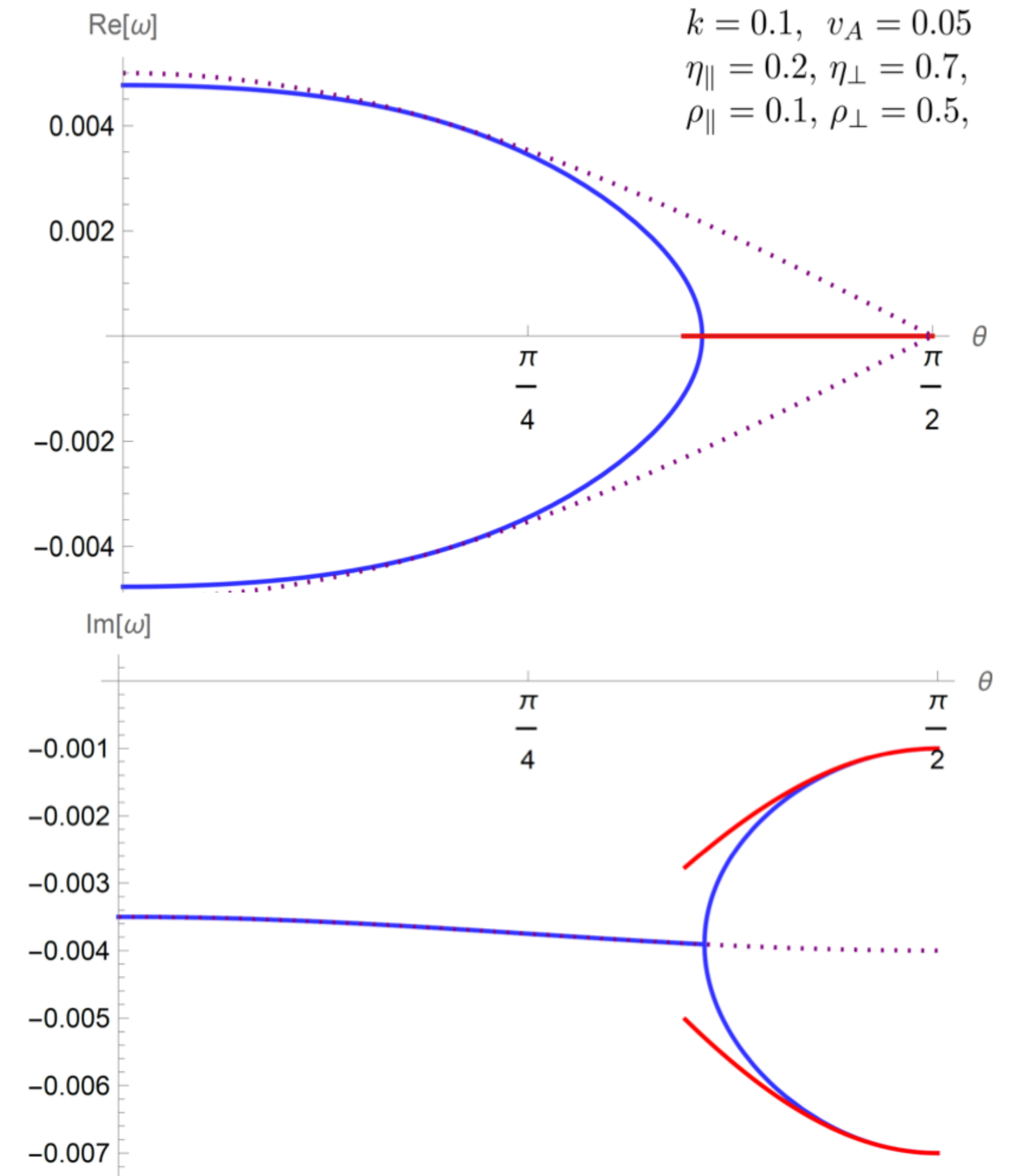
$$\omega = \pm v_A k_{\parallel} - \frac{i}{2}(\tilde{\rho} + \tilde{\eta})k^2 + O\left(\frac{\hat{k}}{|\cos \theta|}k^2\right),$$

for $\hat{k} < |\cos \theta|$,

$$\omega = -ik^2 \rho_{\parallel} - i\left((\rho_{\perp} - \rho_{\parallel})k^2 + \frac{v_A^2}{\eta_{\perp}/(e+p) - \rho_{\parallel}}\right)\cos^2 \theta + O(\cos^4 \theta);$$

$$-ik^2 \frac{\eta_{\perp}}{e+p} - i\left(\frac{1}{e+p}(\eta_{\parallel} - \eta_{\perp})k^2 - \frac{v_A^2}{\frac{\eta_{\perp}}{e+p} - \rho_{\parallel}}\right)\cos^2 \theta + O(\cos^4 \theta)$$

for $\hat{k} > |\cos \theta|$,



1st order MHD analysis

Magneto-sonic sector

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$$\left(M_{(0)} + iM_{(1)} \right) \begin{pmatrix} \delta\epsilon \\ \delta u_x \\ \delta u_z \\ \delta B_x \\ \delta B_z \end{pmatrix} = 0. \quad M_{(0)}, M_{(1)} \text{ not shown for complication}$$

Despite being a univariate quintic equation, we provide a simple method for the solution search that works order-by-order in k . This method serves as a general algorithm applicable to general hydrodynamic equations.

$$\omega = \pm v_1 k - i w_1 k^2 + O(k^3), \quad \omega = \pm v_2 k - i w_2 k^2 + O(k^3), \quad v_{1,2} = \frac{1}{\sqrt{2}} \sqrt{V^2 \pm \sqrt{V^4 - 4v_A^2 c_s^2 \cos^2 \theta}}$$

$$\text{with } V^2 \equiv (c_s^2 + v_A^2) - c_s^2 v_A^2 \sin^2 \theta$$

1st order MHD analysis

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1. In magneto-sonic sector, the singular perturbation problem still exists.

Our remedy also works well.

2. Based on the analytical dispersion relation up to k^2 , we show that

in the rest frame all hydrodynamic modes propagate subluminally,

and ensure stability. But these are insufficient to guarantee covariant causality and covariant stability.

IS-MHD analysis

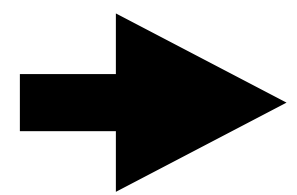
Z.Fang, K. Hattori, J. Hu, *Phys.Rev.D* 112 (2025) 7, 076016

To recover causality, the most common approach is to convert the parabolic nature of the equation into a hyperbolic form.

e.g. $\partial_t u = D \nabla^2 u, \quad u(\mathbf{x}, t) = \frac{1}{(4\pi Dt)^{d/2}} \exp\left(-\frac{|\mathbf{x}|^2}{4Dt}\right)$

with the characteristic line $t = C$

$$\partial_t u + \partial_x J = 0, \quad \tau \partial_t J + J = -D \partial_x u,$$



$$\tau \partial_t^2 u + \partial_t u = D \partial_x^2 u$$

with the characteristic line $x \pm vt = C, \quad v = \sqrt{\frac{D}{\tau}} \leq 1$

IS-MHD analysis

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To recover causality, the most common approach is to convert the parabolic nature of the equation into a hyperbolic form.

diffusion eq $\partial_t u = D \nabla^2 u, \quad \omega = -iDk^2.$ One is not enough.

telegraph eq $\tau \partial_t^2 u + \partial_t u = D \partial_x^2 u$ Two are OK.

$$\omega_{\pm}(k) = -\frac{i}{2\tau} \pm \sqrt{\frac{D}{\tau}k^2 - \frac{1}{4\tau^2}} \xrightarrow{k \rightarrow 0} \begin{cases} \omega_+(k) = -iDk^2 + O(k^4) \\ \omega_-(k) = -i\frac{1}{\tau} + O(k^2) \end{cases}$$

$$\text{Re}\omega_{\pm}(k) \neq 0, \quad \text{if } k > k_{IS} = \frac{1}{2}\sqrt{\frac{1}{D\tau}} \quad \text{UV are propagating.}$$

IS-MHD analysis

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Introducing a relaxation time into the relation between dissipative fluxes and thermodynamic forces, we can analogously construct the Israel-Stewart formulation for MHD (IS-MHD).

Our finding (Alfven modes) : interplay

$$k < k_c(\theta) \quad \text{propagating} \quad \left\{ \begin{array}{l} \omega = \pm v_A k_{\parallel} - \frac{i}{2}(\tilde{\rho} + \tilde{\eta})k^2 + O\left(\frac{\hat{k}}{|\cos \theta|}k^2\right), \\ \text{for } \hat{k} < |\cos \theta|, \end{array} \right.$$

$$k > k_{IS}(\theta) \quad \text{propagating} \qquad k_c(\theta) \leq k \leq k_{IS}(\theta) \quad \text{diffusive window}$$

Summary

1. solve the singular perturbation problem in MHD
2. analytical hydrodynamic dispersion relations up to k^2 ,
3. interplay between relaxation dynamics and singular perturbation in MHD

Outlook

1. diffusive/propagating “phase diagram” in IS-MHD
2. interplay between relaxation dynamics and singular perturbation / on causality and stability ?
3. a causal and stable model for RMHD

Back Up

1. Magnetic moment is an adiabatic invariant

A charged particle gyrates around the magnetic field line. Its magnetic moment is defined as

$$\mu = \frac{mv_{\perp}^2}{2B},$$

where $v_{\perp}$ is the velocity perpendicular to the field and B is the magnetic field strength.

Key physical fact: when the magnetic field varies slowly in space or time (relative to the gyroperiod), μ is an adiabatic invariant:

$$\mu \simeq \text{const.}$$

2. Why does the perpendicular pressure stay unchanged?

Since $\mu = \text{const}$, we have

$$v_{\perp}^2 \propto B.$$

Thus the perpendicular temperature satisfies

$$T_{\perp} \propto B.$$

If the magnetic field is uniform and static ($\partial_t B = 0$ and $\nabla B = 0$), then $T_{\perp}$ does not change. Even when a flux tube is compressed along the field, if its cross-sectional area remains constant,

B. Locality and causality

The Klein-Gordon equation $(\partial_t^2 - \partial_x^2 + m^2)\phi = 0$ is of second order in time, and it can be decomposed into a system of two equations that are of first order in time:

$$\begin{aligned}\partial_t \phi &= \Pi, \\ \partial_t \Pi &= (\partial_x^2 - m^2)\phi.\end{aligned}\tag{S6}$$

Hence, the degrees of freedom are $\Psi = (\phi, \Pi)^T$, and the dynamics of the system has the form $\partial_t \Psi = \mathcal{L}(\partial_x)\Psi$, with

$$\mathcal{L}(ik) = \begin{bmatrix} 0 & 1 \\ -(m^2 + k^2) & 0 \end{bmatrix}.\tag{S7}$$

Clearly, $\mathcal{L}$ is a local operator, being a second-order polynomial in the derivatives. Thus, our Theorem 1 applies, and the Klein-Gordon equation is causal. The general solution to (S6) reads

$$\begin{bmatrix} \phi(t, x) \\ \Pi(t, x) \end{bmatrix} = \int_{-\infty}^{+\infty} \begin{bmatrix} \cos(\omega t) & \omega^{-1} \sin(\omega t) \\ -\omega \sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} \phi(k) \\ \Pi(k) \end{bmatrix} e^{ikx} \frac{dk}{2\pi},\tag{S8}$$

with $\omega = \sqrt{m^2 + k^2}$. Despite the square root, the matrix in (S8) is entire in k , being the exponential of (S7). This is reflected in the fact that the following functions are entire:

$$\begin{aligned}\cos \sqrt{z} &= \sum_{n=0}^{+\infty} \frac{(-1)^n z^n}{(2n)!}, \\ \frac{\sin \sqrt{z}}{\sqrt{z}} &= \sum_{n=0}^{+\infty} \frac{(-1)^n z^n}{(2n+1)!}.\end{aligned}\tag{S9}$$

应用S9的话，所有矩阵元都是entire

C. Positive-frequency solutions cannot be localized

If both degrees of freedom $\phi(0, x)$ and $\Pi(0, x)$ have support inside a compact set $\mathcal{R}$, then both $\phi(k)$ and $\Pi(k)$ are entire functions of $k \in \mathbb{C}$ [29]. On the other hand, the first component of equation (S8) can be decomposed as follows:

$$\phi(t, x) = \int_{-\infty}^{+\infty} \left[\left(\phi(k) + \frac{\Pi(k)}{i\omega} \right) e^{i\omega t} + \left(\phi(k) - \frac{\Pi(k)}{i\omega} \right) e^{-i\omega t} \right] e^{ikx} \frac{dk}{4\pi}.\tag{S10}$$

As we can see, the only way for $\phi(t, x)$ to be a superposition involving only the “positive-frequency waves”, $e^{-i\omega t}$, is to impose $\Pi(k) = -i\omega\phi(k)$. But this contradicts the assumption that both $\Pi(k)$ and $\phi(k)$ are entire, since $\omega = \sqrt{m^2 + k^2}$ has two branch cuts. It follows that compactly-supported solutions are necessarily superpositions of both a positive and a negative frequency part. This superposition results in a perfect cancellation of the superluminal tails of the two individual parts, and this saves causality.

Appendix A: Electric charges in MHD

In neutral plasmas, the electric field $\mathbf{E}$ in the fluid rest frame and the electric charge-density fluctuation δJ^0 are damped out in a finite time scale and are therefore not qualified as hydrodynamic variables [92]. This can be shown in the following way. An electric field $\mathbf{E}$, induced by the charge-density fluctuation, obeys the Gauss's law $\nabla \cdot \mathbf{E} = \delta J^0 / \epsilon_e$ with ϵ_e being a dielectric constant. Charges will be redistributed via an Ohmic current $\mathbf{J} = \sigma \mathbf{E}$. Plugging the Gauss's law into the current conservation $\partial_\mu J^\mu = 0$, we find

$$\frac{\partial \delta J^0(t, \mathbf{x})}{\partial t} = -\nabla \cdot \mathbf{J}(t, \mathbf{x}) = -\frac{\sigma}{\epsilon_e} \delta J^0(t, \mathbf{x}). \quad (\text{A1})$$

Thus, the charge-density fluctuation is damped out in time as

$$\delta J^0(t, \mathbf{x}) = \delta J^0(0, \mathbf{x}) \exp\left(-\frac{\sigma}{\epsilon_e} t\right). \quad (\text{A2})$$

Plugging this solution back to the Gauss's law, we find that the electric field is also damped out in the same time scale.